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LETTER TO THE EDITOR

Operational number-phase probability distribution

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Abstract. An operational probability distribution for number and phase is introduced and the properties are investigated. This operational probability distribution is obtained by replacing the displacement operator of position and momentum in operational phase-space probability distribution with the displacement operator of number and phase. It is shown that the operational number-phase probability distribution is represented by the convolution of two Wigner functions for number and phase.

The phase-space distribution functions for a quantum state of a physical system are useful tools for investigating the dynamical and statistical properties of a quantum mechanical system [1–6]. These include the Glauber–Sudarshan P-function [7–9], the Wigner function [10, 11] and the Husimi Q-function [12, 13] which are closely related to the operator ordering in the mathematical description of a physical system. Except for the Husimi Q-function, the phase-space distribution functions can take negative values due to the non-commutativity or the uncertainty relation of position and momentum. Thus these functions cannot be considered as the probability distributions in the phase space and they are called the quasiprobability distributions. The quasiprobability distributions of position and momentum in the phase space characterize the intrinsic properties of a quantum state of a physical system.

By taking account of the external effects caused by measurement apparatus or by some environmental system, the probability distribution of position and momentum can be defined for a quantum mechanical system [14–19], which is called the operational phase-space probability distribution. The operational phase-space probability distribution, denoted as $\mathcal{W}(r,k)$, is given by

$$\mathcal{W}(r,k) = \frac{1}{2\pi}\,\mathrm{Tr}[\hat{\rho}\hat{D}(r,k)\hat{\sigma}\hat{D}^{\dagger}(r,k)] \tag{1}$$

which is non-negative and normalized as

$$\int_{-\infty}^{\infty}\mathrm{d}r\int_{-\infty}^{\infty}\mathrm{d}k\,\mathcal{W}(r,k) = 1. \tag{2}$$

In equation (1), $\hat{\rho}$ is a statistical operator of the quantum state of the system and $\hat{\sigma}$ is also some statistical operator which is referred to as the quantum ruler [16, 17] or the reference state [18, 19]. The Hilbert space on which these operators are defined is denoted as $\mathcal{H}$. The operator $\hat{D}(r,k)$ induces the displacement in the phase space,

$$\begin{aligned}\hat{D}(r,k) &= \exp[\mathrm{i}(k\hat{x} - r\hat{p})]\\ &= \exp[\mu\hat{a}^{\dagger} - \mu^{*}\hat{a}]\end{aligned} \tag{3}$$

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where $\hat{x}$ and $\hat{p}$ are position and momentum operators and $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators and the complex parameter μ is given by $\mu = (r + \mathrm{i}k)/\sqrt{2}$. In this letter, operators are denoted as a symbol with a hat. When the reference state is the vacuum state, we obtain the Husimi Q-function.

It is shown that the operational phase-space probability distribution is expressed as the convolution of the s-ordered quasiprobability distribution and $(-s)$-ordered quasiprobability distribution [18]. In particular, we have

$$\begin{aligned}\mathcal{W}(r,k) &= \int_{-\infty}^{\infty} \mathrm{d}q \int_{-\infty}^{\infty} \mathrm{d}p \, P(q+r, p+k; \hat{\rho}) Q(q, p; \hat{\sigma}) \\ &= \int_{-\infty}^{\infty} \mathrm{d}q \int_{-\infty}^{\infty} \mathrm{d}p \, Q(q+r, p+k; \hat{\rho}) P(q, p; \hat{\sigma}) \\ &= \int_{-\infty}^{\infty} \mathrm{d}q \int_{-\infty}^{\infty} \mathrm{d}p \, W(q+r, p+k; \hat{\rho}) W(q, p; \hat{\sigma}) \end{aligned} \tag{4}$$

where $P(q, p; \hat{\rho})$, $Q(q, p; \hat{\rho})$ and $W(q, p; \hat{\rho})$ are the Glauber–Sudarshan P-function, the Husimi Q-function and the Wigner function of the quantum state $\hat{\rho}$. Therefore the operational phase-space probability distribution is the smoothed quasiprobability distribution. The smoothing effect is due to the finite accuracy or the sampling of the measurement apparatus [16, 17]. The operational phase-space probability distribution describes the properties of the physical system and the measurement apparatus while the quasiprobability distribution does the properties of only the physical system. The marginal probability distributions are obtained

$$\mathcal{W}(r) = \int_{-\infty}^{\infty} \mathrm{d}k \mathcal{W}(r,k) = \int_{-\infty}^{\infty} \mathrm{d}q \, f(q-r) \langle q|\hat{\rho}|q\rangle \tag{5}$$

$$\mathcal{W}(k) = \int_{-\infty}^{\infty} \mathrm{d}r \mathcal{W}(r,k) = \int_{-\infty}^{\infty} \mathrm{d}p \, g(k-p) \langle p|\hat{\rho}|p\rangle \tag{6}$$

where $|q\rangle$ and $|p\rangle$ are the position and momentum eigenstates and $f(q) = \langle q|\hat{\sigma}|q\rangle$ and $g(p) = \langle p|\hat{\sigma}|p\rangle$ represents are the filter functions of the measurement apparatus. The properties of the operational phase-space probability distribution have been investigated in detail [14–19]. It is shown that the operational phase-space probability distribution can describe the realistic optical measurement [20, 21] and the simultaneous measurement of position and momentum [22, 23]. Furthermore we remark that the operational phase-space probability distributions are closely related to the stochastic or fuzzy-space formulation of quantum mechanics [24, 25].

The operational phase-space probability distribution is a function of position and momentum which are canonically conjugate to each other, and the properties have been investigated in detail. Number and phase are also an important canonical pair in quantum optical systems. Therefore the purpose of this letter is to introduce the operational probability distribution for number and phase and to investigate the properties. To obtain the operational number-phase probability distribution, let us rewrite the operational phase-space probability distribution in another form [18]. We first introduce an auxiliary Hilbert space $\mathcal{H}_a$ and we define a statistical operator $\hat{\sigma}_\mathrm{a}$ in this Hilbert space by

$$\hat{\sigma}_\mathrm{a} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |m_\mathrm{a}\rangle [\langle n|\hat{\sigma}|m\rangle] \langle n_\mathrm{a}| \tag{7}$$

where $|n\rangle$ and $|n_\mathrm{a}\rangle$ are the Fock states in the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_\mathrm{a}$. We denote all of the quantities in the auxiliary Hilbert space $\mathcal{H}_\mathrm{a}$ by adding the subscript ‘a’. The statistical operator $\hat{\sigma}_a$ satisfies the relations $\langle m_a|\hat{\sigma}_a|n_a\rangle = \langle n|\hat{\sigma}|m\rangle$ and $\langle x_a|\hat{\sigma}_a|y_a\rangle = \langle y|\hat{\sigma}|x\rangle$. Using the statistical operator $\hat{\sigma}_a$, we can express the operational phase-space probability distribution $\mathcal{W}(r,k)$ given by equation (1) in the following form [18]:

$$\mathcal{W}(r,k) = \langle \psi(r,k)|\hat{\rho}\otimes\hat{\sigma}_a|\psi(r,k)\rangle \tag{8}$$

where $|\psi(r,k)\rangle$ is a state vector in the tensor product Hilbert space $\mathcal{H}\otimes\mathcal{H}_a$,

$$|\psi(r,k)\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}q\, |r+q\rangle \otimes |q_a\rangle \mathrm{e}^{\mathrm{i}kq}. \tag{9}$$

It is easy to see that this state vector is the simultaneous eigenstate of the relative-position operator $\hat{x}\otimes\hat{1}_a - \hat{1}\otimes\hat{x}_a$ and the momentum-sum operator $\hat{p}\otimes\hat{1}_a + \hat{1}\otimes\hat{p}_a$ with respective eigenvalues r and k, where $\hat{1}$ and $\hat{1}_a$ are identity operators defined on the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_a$. Furthermore the set $\{|\psi(r,k)\rangle \mid r,k\in\mathbb{R}\}$ becomes complete orthonormal system in the Hilbert space $\mathcal{H}\otimes\mathcal{H}_a$,

$$\langle \psi(r,k)|\psi(r',k')\rangle = \delta(r-r')\delta(k-k') \tag{10}$$

$$\int_{-\infty}^{\infty}\mathrm{d}r\int_{-\infty}^{\infty}\mathrm{d}k\, |\psi(r,k)\rangle\langle\psi(r,k)| = \hat{1}\otimes\hat{1}_a. \tag{11}$$

We remark that the projection operator $|\psi(r,k)\rangle\langle\psi(r,k)|$ in the tensor product Hilbert space $\mathcal{H}\otimes\mathcal{H}_a$ is nothing but the Naimark extension of the positive operator-valued measure $(2\pi)^{-1}\hat{D}(r,k)\hat{\sigma}\hat{D}^\dagger(r,k)$ in the original Hilbert space $\mathcal{H}$, and the statistical operator $\hat{\sigma}_a$ in the Hilbert space $\mathcal{H}_a$ is the Naimark state [26].

We now consider number and phase variables. Since the phase and number variables, ϕ and n, correspond respectively to the position and momentum variables, q and p, a state vector $|\psi(\phi,n)\rangle$ corresponding to the state vector $|\psi(r,k)\rangle$ is given by

$$|\psi(\phi,n)\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\mathrm{d}\varphi\, |\phi+\varphi\rangle\otimes|\varphi_a\rangle\mathrm{e}^{\mathrm{i}n\varphi} \tag{12}$$

where $|\phi\rangle$ is the eigenstate of the Susskind–Glogower phase operator $\hat{E} = \sum_{n=0}^{\infty}|n\rangle\langle n+1|$ [27, 28] which is isometric but not unitary,

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}|n\rangle\mathrm{e}^{-\mathrm{i}\phi n}. \tag{13}$$

Because of the non-unitarity of the Susskind–Glogower phase operator $\hat{E}$, we have $\hat{E}|\phi\rangle = \mathrm{e}^{-\mathrm{i}\phi}|\phi\rangle$ but not $\hat{E}^\dagger|\phi\rangle = \mathrm{e}^{\mathrm{i}\phi}|\phi\rangle$. In this letter, we restrict the range of the phase variable ϕ to be $-\pi \leqslant \phi < \pi$. The set of the Susskind–Glogower phase eigenstates $\{|\phi\rangle \mid -\pi\leqslant\phi<\pi\}$ becomes an overcomplete system in the Hilbert space $\mathcal{H}$, which satisfies

$$\langle\phi|\varphi\rangle = \vartheta(\phi-\varphi) \qquad \int_{-\pi}^{\pi}\mathrm{d}\phi\, |\phi\rangle\langle\phi| = \hat{1} \tag{14}$$

where the function $\vartheta(\phi)$ is defined by

$$\vartheta(\phi) = \frac{1}{4\pi} + \frac{1}{2}\delta(\phi) - \frac{\mathrm{i}}{4\pi}\cot\left(\frac{\phi}{2}\right). \tag{15}$$

It is found from equations (12)–(15) that the set of the state vectors $|\psi(\phi,n)\rangle$ becomes an overcomplete system in the tensor product Hilbert space $\mathcal{H}\otimes\mathcal{H}_a$,

$$\langle\psi(\phi,n)|\psi(\phi',n')\rangle = \delta_{n,n'}\vartheta_n(\phi-\phi') \tag{16}$$

$$\sum_{n=0}^{\infty}\int_{-\pi}^{\pi}\mathrm{d}\phi\, |\psi(\phi,n)\rangle\langle\psi(\phi,n)| = \hat{1}\otimes\hat{1}_a \tag{17}$$

where the function $\vartheta_n(\phi)$ is given by

$$\vartheta_n(\phi) = \frac{1 - e^{i(n+1)\phi}}{2\pi(1 - e^{i\phi})}. \tag{18}$$

It is easy to see that the state vector $|\psi(\phi, n)\rangle$ is the eigenstate of the number-sum operator $\hat{a}^\dagger\hat{a} \otimes \hat{1}_a + \hat{1} \otimes \hat{a}_a^\dagger\hat{a}_a$ with eigenvalue n. If the Susskind–Glogower phase operator $\hat{E}$ was unitary, the state vector $|\psi(\phi, n)\rangle$ would have been the eigenstate of the phase-difference operator $\hat{E} \otimes \hat{E}_a^\dagger \sim e^{-i(\hat{\phi}\otimes\hat{1}_a - \hat{1}\otimes\hat{\phi}_a)}$ with eigenvalue $e^{-i\phi}$.

We now introduce a normalizable and non-negative function $\mathcal{W}(\phi, n)$ of number and phase by

$$\begin{aligned}\mathcal{W}(\phi, n) &= \langle\psi(\phi, n)|\hat{\rho} \otimes \hat{\sigma}_a|\psi(\phi, n)\rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi \int_{-\pi}^{\pi} d\varphi'\, e^{in(\varphi-\varphi')}\langle\varphi' + \phi|\hat{\rho}|\phi + \varphi\rangle\langle\varphi_a'|\hat{\sigma}_a|\varphi_a\rangle\end{aligned} \tag{19}$$

where $\hat{\rho}$ is a quantum state of the system in the Hilbert space $\mathcal{H}$ and $\hat{\sigma}_a$ is a non-negative operator with $\mathrm{Tr}_a\,\hat{\sigma}_a = 1$ in the auxiliary Hilbert space $\mathcal{H}_a$. The completeness relation of the state vector $|\psi(\phi, n)\rangle$ yields the normalization condition

$$\sum_{n=0}^{\infty}\int_{-\pi}^{\pi} d\phi\, \mathcal{W}(\phi, n) = 1. \tag{20}$$

Note that this function is defined in terms of the state vectors in the tensor product Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$. Hence we rewrite the function $\mathcal{W}(\phi, n)$ so that it can be defined in the original Hilbert space $\mathcal{H}$. To this end, we define a statistical operator $\hat{\sigma}$ in the Hilbert space $\mathcal{H}$ by

$$\hat{\sigma} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\hat{T}^\dagger|m\rangle[\langle n_a|\hat{\sigma}_a|m_a\rangle]\langle n|\hat{T} \tag{21}$$

where the anti-unitary operator $\hat{T}$ takes the complex conjugate of a state vector on which it acts, that is,

$$|\psi\rangle = \sum_{n=0}^{\infty} a_n|n\rangle \longrightarrow \hat{T}|\psi\rangle = |\psi^*\rangle \equiv \sum_{n=0}^{\infty} a_n^*|n\rangle. \tag{22}$$

Then since we have the relation $\langle\varphi|\hat{\sigma}|\varphi'\rangle = \langle\varphi_a'|\hat{\sigma}_a|\varphi_a\rangle$, the function $\mathcal{W}(\phi, n)$ becomes

$$\mathcal{W}(\phi, n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi \int_{-\pi}^{\pi} d\varphi'\, e^{in(\varphi-\varphi')}\langle\varphi' + \phi|\hat{\rho}|\phi + \varphi\rangle\langle\varphi|\hat{\sigma}|\varphi'\rangle. \tag{23}$$

Furthermore using the relations $|\phi + \varphi\rangle = e^{-i\phi\hat{n}}|\varphi\rangle$ and $e^{-in\varphi}|\varphi\rangle = \hat{E}^n|\varphi\rangle$, where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the number operator, we obtain the following expression for the function $\mathcal{W}(\phi, n)$,

$$\begin{aligned}\mathcal{W}(\phi, n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi \int_{-\pi}^{\pi} d\varphi'\, \langle\varphi'|e^{i\phi\hat{n}}\hat{\rho}e^{-i\phi\hat{n}}|\varphi\rangle\langle\varphi|\hat{E}^{\dagger n}\hat{\sigma}\hat{E}^n|\varphi'\rangle \\ &= \frac{1}{2\pi}\mathrm{Tr}[\hat{\rho}\hat{D}(\phi, n)\hat{\sigma}\hat{D}^\dagger(\phi, n)]\end{aligned} \tag{24}$$

where the operator $\hat{D}(\phi, n)$ is defined by

$$\hat{D}(\phi, n) = e^{-i\phi\hat{n}}\hat{E}^{\dagger n}e^{\frac{1}{2}in\phi} = \hat{E}^{\dagger n}e^{-i\phi\hat{n}}e^{-\frac{1}{2}in\phi} \tag{25}$$

which is the displacement operator of number and phase. Because of the non-unitarity of the Susskind–Glogower phase operator $\hat{E}$, the operator $\hat{D}(\phi, n)$ is isometric but not unitary and satisfies the relations

$$\hat{D}(\phi, n)\hat{D}(\phi', n') = \hat{D}(\phi + \phi', n + n')\mathrm{e}^{\frac{1}{2}\mathrm{i}(n\phi' - n'\phi)} \tag{26}$$

$$\hat{D}^{\dagger}(\phi, n)\hat{D}(\phi, n) = \hat{1} \tag{27}$$

$$\hat{D}(\phi, n)\hat{D}^{\dagger}(\phi, n) = \hat{1} - \sum_{k=0}^{n-1} |k\rangle\langle k| \tag{28}$$

where we have used the relations $\hat{E}\hat{E}^{\dagger} = \hat{1}$ and $\hat{E}^{\dagger}\hat{E} = \hat{1} - |0\rangle\langle 0|$. It was shown that the set of the number-phase displacement operators, $\mathcal{S} = \{\hat{D}(\phi, n) | -\pi \leqslant \phi < \pi,\ n = 0, 1, 2, \ldots\}$, becomes the Weyl semigroup for number and phase [29]. If there was a Hermitian phase operator $\hat{\phi}$ canonically conjugate to the number operator $\hat{n}$, we would have obtained the expression $\hat{D}(\phi, n) = \exp[\mathrm{i}(n\hat{\phi} - \phi\hat{n})]$ for the number-phase displacement operator and the set $\mathcal{S}$ would have been the Weyl group for number and phase. Note that the function $\mathcal{W}(\phi, n)$ has the same form of the operational phase-space probability distribution $\mathcal{W}(r, k)$ and the only difference between them is the displacement operator; the position-momentum displacement operator $\hat{D}(r, k)$ is used for $\mathcal{W}(r, k)$ and the number-phase displacement operator $\hat{D}(\phi, n)$ for $\mathcal{W}(\phi, n)$. Therefore we can show that the function $\mathcal{W}(\phi, n)$ has the meaning of the operational number-phase probability distribution. For this reason, we refer to the function $\mathcal{W}(\phi, n)$ as the operational number-phase probability distribution.

We now consider the properties of the operational number-phase probability distribution $\mathcal{W}(\phi, n)$ given by equation (24). The marginal distributions becomes

$$\mathcal{W}(n) = \int_{-\pi}^{\pi} \mathrm{d}\phi\, \mathcal{W}(\phi, n) = \sum_{m=0}^{n} \mu(n - m)\langle m|\hat{\rho}|m\rangle \tag{29}$$

$$\mathcal{W}(\phi) = \sum_{n=0}^{\infty} \mathcal{W}(\phi, n) = \int_{-\pi}^{\pi} \mathrm{d}\varphi\, \nu(\phi - \varphi)\langle\varphi|\hat{\rho}|\varphi\rangle \tag{30}$$

where the functions $\mu(n) = \langle n|\hat{\sigma}|n\rangle$ and $\nu(\phi) = \langle\phi|\hat{\sigma}|\phi\rangle$ are considered the filter functions of the measurement apparatus in the number and phase measurement. The filter functions determine the measurement accuracy. For example, when the measurement apparatus is in the vacuum state which corresponds to $\hat{\sigma} = |0\rangle\langle 0|$, the filter functions becomes $\mu(n) = \delta_{0,n}$ and $\nu(\phi) = (2\pi)^{-1}$. Then we obtain the marginal probability distributions $\mathcal{W}(n) = \langle n|\hat{\rho}|n\rangle$ and $\mathcal{W}(\phi) = (2\pi)^{-1}$. This result indicates that we cannot measure the phase of the physical system by means of the measurement apparatus in the quantum state with completely uncertain phase. Using the marginal distributions $\mathcal{W}(n)$ and $\mathcal{W}(\phi)$, we obtain the operational characteristic functions for number and phase,

$$\mathcal{F}_n(x) = \sum_{n=0}^{\infty} \mathrm{e}^{-\mathrm{i}nx}\mathcal{W}(n) = F_n(x; \hat{\rho})F_n(x; \hat{\sigma}) \tag{31}$$

$$\mathcal{F}_{\phi}(x) = \int_{-\pi}^{\pi} \mathrm{d}\phi\, \mathrm{e}^{-\mathrm{i}\phi x}\mathcal{W}(\phi) = F_{\phi}(x; \hat{\rho})F_{\phi}(x; \hat{\sigma}) \tag{32}$$

where $F_n(x; \hat{\rho})$ and $F_{\phi}(x; \hat{\rho})$ are the intrinsic number and phase characteristic functions of the quantum state $\hat{\rho}$,

$$F_n(x; \hat{\rho}) = \mathrm{Tr}[\mathrm{e}^{-\mathrm{i}\hat{n}x}\hat{\rho}] \qquad F_{\phi}(x; \hat{\rho}) = \mathrm{Tr}[\hat{E}(x)\hat{\rho}]. \tag{33}$$

Here we set $\hat{E}(x) = \int_{-\pi}^{\pi} \mathrm{d}\phi\, |\phi\rangle\mathrm{e}^{-\mathrm{i}\phi x}\langle\phi|$. Note that if x is a non-negative (or negative) integer, the equality $\hat{E}(x) = \hat{E}^{x}$ (or $\hat{E}(x) = \hat{E}^{\dagger|x|}$) holds. The operational characteristic functions yield the operational moments of number and phase,

$$\langle \hat{n}^m \rangle_{\rm op} = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \langle \hat{n}^k \rangle_{\hat{\rho}} \langle \hat{n}^{m-k} \rangle_{\hat{\sigma}} \tag{34}$$

$$\langle \hat{\phi}^m \rangle_{\rm op} = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \langle \hat{\phi}^k \rangle_{\hat{\rho}} \langle \hat{\phi}^{m-k} \rangle_{\hat{\sigma}} \tag{35}$$

where we set $\langle \hat{n}^k \rangle_{\hat{\rho}} = {\rm Tr}[\hat{n}^k \hat{\rho}]$ and $\langle \hat{\phi}^k \rangle_{\hat{\rho}} = \int_{-\pi}^{\pi} {\rm d}\phi\, \phi^k \langle \phi | \hat{\rho} | \phi \rangle$. In particular, we obtain

$$\langle \Delta \hat{n}^n \rangle_{\rm op} = \langle (\hat{n} - \langle \hat{n} \rangle_{\rm op})^2 \rangle_{\rm op} = \langle \Delta \hat{n}^2 \rangle_{\hat{\rho}} + \langle \Delta \hat{n}^2 \rangle_{\hat{\sigma}} \tag{36}$$

$$\langle \Delta \hat{\phi}^n \rangle_{\rm op} = \langle (\hat{\phi} - \langle \hat{\phi} \rangle_{\rm op})^2 \rangle_{\rm op} = \langle \Delta \hat{\phi}^2 \rangle_{\hat{\rho}} + \langle \Delta \hat{\phi}^2 \rangle_{\hat{\sigma}} \tag{37}$$

which clearly shows the enhancement of the number and phase fluctuations that is caused by the measurement apparatus.

We next consider the relation between the operational number-phase probability distribution $\mathcal{W}(\phi, n)$ and the Wigner function for number and phase. The number-phase Wigner function can be constructed within the framework of the Pegg–Barnett phase operator formalism [30, 31]. The Pegg–Barnett phase operator $\hat{\phi}_s$ and its eigenstate $|\phi_m\rangle$ are defined in a $(s+1)$-dimensional Hilbert space $\mathcal{H}_{s+1}$,

$$\hat{\phi}_s = \sum_{m=0}^{s} |\phi_m\rangle \phi_m \langle \phi_m| \tag{38}$$

$$|\phi_m\rangle = \frac{1}{\sqrt{1+s}} \sum_{n=0}^{s} {\rm e}^{-{\rm i}n\phi_m} |n\rangle \tag{39}$$

with $\phi_m = -\pi + 2\pi m/(s+1) \equiv -\pi + \Delta m$. The exponential $\hat{E}_s = \exp(-{\rm i}\hat{\phi}_s)$ of the Pegg–Barnett phase operator is a unitary operator,

$$\begin{aligned} \hat{E}_s &= \sum_{m=0}^{s} |\phi_m\rangle {\rm e}^{-{\rm i}\phi_m} \langle \phi_m| \\ &= \sum_{n=0}^{s-1} |n\rangle\langle n+1| + {\rm e}^{{\rm i}(1+s)\pi} |s\rangle\langle 0|. \end{aligned} \tag{40}$$

The set of the Pegg–Barnett phase eigenstates $\{|\phi_m\rangle \,|\, m = 0, 1, \ldots, s\}$ spans a complete orthonormal system in the Hilbert space $\mathcal{H}_{s+1}$. Taking the limit $s \to \infty$ after all the calculations are complete yields the physical quantities such average value and fluctuation. The number-phase Wigner function $W_s(\phi_m, n; \hat{\rho})$ in the Pegg–Barnett phase operator formalism [32] is a discrete Wigner function [33],

$$W_s(\phi_m, n; \hat{\rho}) = \frac{1}{1+s} \sum_{k=0}^{s} {\rm e}^{2{\rm i}\Delta k n} \langle \phi_{m-k} | \hat{\rho} | \phi_{m+k} \rangle \tag{41}$$

which is a quasiprobability distribution since it can take negative values. The discrete number-phase Wigner function $W_s(\phi_m, n; \hat{\rho})$ is normalized as

$$\sum_{m=0}^{s} \sum_{n=0}^{s} W_s(\phi_m, n; \hat{\rho}) = 1. \tag{42}$$

It is easily seen from the definition that the following relations are satisfied

$$\sum_{n=0}^{s} W_s(\phi_m, n; \hat{\rho}) = \langle \phi_m | \hat{\rho} | \phi_m \rangle \stackrel{s\to\infty}{\longrightarrow} \langle \phi | \hat{\rho} | \phi \rangle \tag{43}$$

$$\sum_{m=0}^{s} W_s(\phi_m, n; \hat{\rho}) = \langle n|\hat{\rho}|n\rangle. \tag{44}$$

The properties of the number-phase Wigner function in the Pegg–Barnett phase operator formalism have been investigated in detail [32].

To find the relation between the operational number-phase probability distribution $\mathcal{W}(\phi, n)$ and the discrete number-phase Wigner function $W_s(\phi_m, n; \hat{\rho})$, let us calculate the convolution of the two discrete number-phase Wigner functions $W_s(\phi_m, n; \hat{\rho})$ and $W_s(\phi_m, n; \hat{\sigma})$;

$$\begin{aligned}
\sum_{k=0}^{s}\sum_{l=0}^{s} & W_s(\phi_{m+k}, n+l; \hat{\rho}) W_s(\phi_k, l; \hat{\sigma}) \\
&= \frac{1}{1+s}\sum_{k=0}^{s}\sum_{l=0}^{s} \mathrm{e}^{2\mathrm{i}\Delta l n}\langle\phi_{m+k-l}|\hat{\rho}|\phi_{m+k+l}\rangle\langle\phi_{k+l}|\hat{\sigma}|\phi_{k-l}\rangle \\
&= \frac{1}{1+s}\sum_{k=0}^{s}\sum_{l=0}^{s} \langle\phi_{m+k-l}|\hat{E}_s^n\hat{\rho}\hat{E}_s^{\dagger n}|\phi_{m+k+l}\rangle\langle\phi_{k+l}|\hat{\sigma}|\phi_{k-l}\rangle
\end{aligned} \tag{45}$$

where we have used the eigenvalue equations of the Pegg–Barnett phase operator,

$$\hat{E}_s|\phi_m\rangle = \mathrm{e}^{-\mathrm{i}\phi_m}|\phi_m\rangle \qquad \hat{E}_s^{\dagger}|\phi_m\rangle = \mathrm{e}^{\mathrm{i}\phi_m}|\phi_m\rangle. \tag{46}$$

Since the Pegg–Barnett phase eigenstate satisfies the relation

$$|\phi_{m+k}\rangle = \mathrm{e}^{-\mathrm{i}\Delta k\hat{n}}|\phi_m\rangle \tag{47}$$

we can further calculate equation (45) as follows

$$\begin{aligned}
\sum_{k=0}^{s}\sum_{l=0}^{s} & W_s(\phi_{m+k}, n+l; \hat{\rho}) W_s(\phi_k, l; \hat{\sigma}) \\
&= \frac{1}{1+s}\sum_{k=0}^{s}\sum_{l=0}^{s} \langle\phi_{k-l}|\mathrm{e}^{\mathrm{i}\Delta m\hat{n}}\hat{E}_s^n\hat{\rho}\hat{E}_s^{\dagger n}\mathrm{e}^{-\mathrm{i}\Delta m\hat{n}}|\phi_{k+l}\rangle\langle\phi_{k+l}|\hat{\sigma}|\phi_{k-l}\rangle \\
&= \frac{1}{1+s}\sum_{k=0}^{s}\sum_{l=0}^{s} \langle\phi_{k-l}|\mathrm{e}^{\mathrm{i}(\phi_m+\pi)\hat{n}}\hat{E}_s^n\hat{\rho}\hat{E}_s^{\dagger n}\mathrm{e}^{-\mathrm{i}(\phi_m+\pi)\hat{n}}|\phi_{k+l}\rangle\langle\phi_{k+l}|\hat{\sigma}|\phi_{k-l}\rangle \\
&= \frac{1}{1+s}\mathrm{Tr}_s[\hat{\rho}\hat{D}_s(\phi_m+\pi)\hat{\sigma}\hat{D}_s^{\dagger}(\phi_m+\pi, n)]
\end{aligned} \tag{48}$$

where Tr_s means the trace operation over the Hilbert space $\mathcal{H}_{s+1}$ and the operator $\hat{D}_s(\phi_m, n)$ induces the number-phase displacement in the Hilbert space $\mathcal{H}_{s+1}$,

$$\hat{D}_s(\phi_m, n) = \mathrm{e}^{-\mathrm{i}\phi_m\hat{n}}\hat{E}_s^{\dagger n}\mathrm{e}^{\frac{1}{2}\mathrm{i}n\phi_m} = \hat{E}_s^{\dagger n}\mathrm{e}^{-\mathrm{i}\phi_m\hat{n}}\mathrm{e}^{-\frac{1}{2}\mathrm{i}n\phi_m} \tag{49}$$

which corresponds to the number-phase displacement operator $\hat{D}(\phi, n)$ in the limit $s \to \infty$. Therefore we have found the following relation between the operational number-phase probability distribution $\mathcal{W}(\phi, n)$ and the discrete number-phase Wigner function,

$$\mathcal{W}(\phi+\pi, n) = \lim_{s\to\infty}\frac{1+s}{2\pi}\sum_{k=0}^{s}\sum_{l=0}^{s} W_s(\phi_{m+k}, n+l; \hat{\rho}) W_s(\phi_k, l; \hat{\sigma}) \tag{50}$$

which indicates that the operational number-phase probability distribution $\mathcal{W}(\phi, n)$ is equivalent to the convolution of the two discrete number-phase Wigner functions in the Pegg–Barnett phase operator formalism. Note that the function $\mathcal{W}(\phi+\pi, n)$ but not $\mathcal{W}(\phi, n)$ appears on the left-hand side of equation (50) since we take the convolution of the Wigner functions $W_s(\phi_{m+k}, n+l; \hat{\rho})$ and $W_s(\phi_k, l; \hat{\sigma})$ but not $W_s(\phi_m+\phi_k, n+l; \hat{\rho})$ and $W_s(\phi_k, l; \hat{\sigma})$. The difference between ϕ_{m+k} and $\phi_m + \phi_k$ is just π radian in our definition of the phase range.

We next introduce a normalizable function $W(\phi, n; \hat{\rho})$ of number and phase for the quantum state $\hat{\rho}$ [34],

$$W(\phi, n; \hat{\rho}) = \frac{1}{2\pi}\left\{\langle n|\hat{\rho}|n\rangle + \sum_{m=1}^{\infty}[\mathrm{e}^{\mathrm{i}m\phi}\langle m+n|\hat{\rho}|n\rangle + (\text{c.c.})]\right\} \tag{51}$$

where (c.c.) means taking the complex conjugate of the first term in the square bracket. This function takes negative values and the normalization condition is given by

$$\sum_{n=0}^{\infty}\int_{-\pi}^{\pi}\mathrm{d}\phi\, W(\phi, n; \hat{\rho}) = 1. \tag{52}$$

It is easy to see that the function $W(\phi, n)$ satisfies the following relations:

$$\int_{-\pi}^{\pi}\mathrm{d}\phi\, W(\phi, n; \hat{\rho}) = \langle n|\hat{\rho}|n\rangle \tag{53}$$

$$\sum_{n=0}^{\infty} W(\phi, n; \hat{\rho}) = \langle\phi|\hat{\rho}|\phi\rangle \tag{54}$$

$$\sum_{n=0}^{\infty}\int_{-\pi}^{\pi}\mathrm{d}\phi\, W(\phi, n; \hat{\rho})W(\phi, n; \hat{\sigma}) = \frac{1}{2\pi}\,\mathrm{Tr}[\hat{\rho}\hat{\sigma}]. \tag{55}$$

Therefore we find that the function $W(\phi, n; \hat{\rho})$ has the same properties as those of the Wigner function for number and phase. Hence, we refer to this function as the number-phase Wigner function. To obtain the relation between the operational number-phase probability distribution $\mathcal{W}(\phi, n)$ and the number-phase Wigner function $W(\phi, n; \hat{\rho})$, we calculate the convolution of the two number-phase Wigner function $W(\phi, n; \hat{\rho})$ and $W(\phi, n; \hat{\sigma})$,

$$\begin{aligned}
&\sum_{m=0}^{\infty}\int_{-\pi}^{\pi}\mathrm{d}\psi\, W(\phi+\psi, n+m; \hat{\rho})W(\psi, m; \hat{\sigma}) \\
&\quad = \frac{1}{2\pi}\sum_{m=0}^{\infty}\Bigg\{\langle m+n|\hat{\rho}|m+n\rangle\langle m|\hat{\sigma}|m\rangle \\
&\qquad + \sum_{k=1}^{\infty}[\mathrm{e}^{\mathrm{i}k\phi}\langle m+n+k|\hat{\rho}|m+n\rangle\langle m|\hat{\sigma}|m+k\rangle + (\text{c.c.})]\Bigg\} \\
&\quad = \frac{1}{2\pi}\sum_{m=0}^{\infty}\Bigg\{\langle m|\hat{D}^{\dagger}(\phi, n)\hat{\rho}\hat{D}(\phi, n)|m\rangle\langle m|\hat{\sigma}|m\rangle \\
&\qquad + \sum_{k=1}^{\infty}[\langle m+k|\hat{D}^{\dagger}(\phi, n)\hat{\rho}\hat{D}(\phi, n)|m\rangle\langle m|\hat{\sigma}|m+k\rangle + (\text{c.c.})]\Bigg\} \\
&\quad = \frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\langle m|\hat{D}^{\dagger}(\phi, n)\hat{\rho}\hat{D}(\phi, n)|k\rangle\langle k|\hat{\sigma}|m\rangle \\
&\quad = \frac{1}{2\pi}\,\mathrm{Tr}[\hat{D}^{\dagger}(\phi, n)\hat{\rho}\hat{D}(\phi, n)\hat{\sigma}].
\end{aligned} \tag{56}$$

Therefore the operational number-phase probability distribution $\mathcal{W}(\phi, n)$ is represented by the convolution of the number-phase Wigner functions of the quantum state $\hat{\rho}$ and the reference state $\hat{\sigma}$,

$$\mathcal{W}(\phi, n) = \sum_{m=0}^{\infty} \int_{-\pi}^{\pi} \mathrm{d}\varphi\, W(\phi + \varphi, n + m; \hat{\rho}) W(\varphi, m; \hat{\sigma}). \tag{57}$$

It is found from equations (50) and (57) that the operational number-phase probability distribution $\mathcal{W}(\phi, n)$ is the smoothed number-phase Wigner function. The smoothing effect is due to the finite accuracy of the apparatus in the number and phase measurement.

There are two other number-phase Wigner functions that satisfy the relations given by equations (52)–(55) [35–37]. One, denoted as $S(\phi, n; \hat{\rho})$ [35, 36], is given by

$$S(\phi, n; \hat{\rho}) = \frac{1}{2\pi} \Bigg\{ \langle n|\hat{\rho}|n\rangle + \sum_{m=1}^{n} [\mathrm{e}^{\mathrm{i}(2m-1)\phi} \langle n+m-1|\hat{\rho}|n-m\rangle + (\text{c.c.})] \\ + \sum_{m=1}^{n} [\mathrm{e}^{2\mathrm{i}m\phi} \langle n+m|\hat{\rho}|n-m\rangle + (\text{c.c.})] \Bigg\} \tag{58}$$

and the other number-phase Wigner function $\tilde{S}(\phi, n; \hat{\rho})$ [37] is related to the number-phase Wigner function $S(\phi, n; \hat{\rho})$ by $S(\phi, n; \hat{\rho}) = \tilde{S}(\phi, n; \hat{\rho}) + \tilde{S}(\phi, n - \frac{1}{2}; \hat{\rho})$. Although the number-phase Wigner functions $S(\phi, n; \hat{\rho})$ and $W(\phi, n; \hat{\rho})$ are quite different, they exhibit similar properties. By straightforward calculation, we can show that the operational number-phase probability distribution $\mathcal{W}(\phi, n)$ is represented by the convolution of the two number-phase Wigner functions $S(\phi, n; \hat{\rho})$ and $S(\phi, n; \hat{\sigma})$,

$$\mathcal{W}(\phi, n) = \sum_{m=0}^{\infty} \int_{-\pi}^{\pi} \mathrm{d}\varphi\, S(\phi + \varphi, n + m; \hat{\rho}) S(\varphi, m; \hat{\sigma}). \tag{59}$$

All the results obtained for the function $\mathcal{W}(\phi, n)$ of number and phase given by equation (24) indicate that this function is the operational number-phase probability distribution of the physical system. Therefore we have obtained the operational number-phase probability distribution $\mathcal{W}(\phi, n)$ and investigated the properties. Since the operator $\hat{\mathcal{X}}(\phi, n) = (2\pi)^{-1} \hat{D}(\phi, n) \hat{\sigma} \hat{D}^{\dagger}(\phi, n)$ is a positive operator-valued measure that satisfies $\hat{\mathcal{X}}(\phi, n) \geqslant 0$ and $\sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \mathrm{d}\phi\, \hat{\mathcal{X}}(\phi, n) = \hat{1}$, we can consider operational number and phase observables in the same way as that for the operational position and momentum observables [18, 20].

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